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Two new theorems on integral inequalities involving maximums and minimums of ratios

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Abstract

In this article, we present two new theorems relating to the upper bounds of specific two-dimensional integral inequalities. The first theorem is based on the maximums of the ratios of the two variables, while the second is based on the minimums. The obtained bounds are quite manageable, with constant factors defined by simple integrals. Complete proofs are provided for the sake of rigor. To illustrate the scope and implications of these results, we present several examples, some of which focus on the standard beta and gamma functions.

Keywords: Two-dimensional integral inequalities; Hardy-Hilbert-type integral inequalities; beta function; gamma function.

MSC2020: 26D15, 33E20

1 Introduction

There is a wide variety of integral inequalities. These powerful tools can be used to estimate integral values, bound functions and analyse the behavior of solutions to differential equations. In two dimensions, the most popular are the Hardy-Hilbert type integral inequalities. Numerous modifications and generalizations of these inequalities have been developed over time. See, for example, [4, 2, 13, 14]. Certain modified Hardy-Hilbert-type integral inequalities involve minimums or maximums of the variables or certain transformations of these variables. These formulations allow for more flexible and precise characterizations of functional relationships, particularly in settings where asymmetry or variable dominance plays a significant role. They have inspired a substantial and continually expanding body of literature. See, for example, [12, 5, 1, 6, 8, 9, 10, 11, 7, 3].

In this article, we make a contribution to the study of two-dimensional integral inequalities by presenting two new theorems. The first of these is innovative in that it depends on certain maximums of the ratios of the variables, specifically

$$\max\left(\frac{x}{y}, \frac{y}{x}\right), \quad \max\left(\frac{x^2}{y^2}, 1\right),$$

where x and y denote the two main variables. It also depends on three functions: f, g and S. The first two are the main functions of interest, while the third, S, is an auxiliary function that can be adjusted according to the mathematical context. The second theorem presents an analogous result involving the corresponding minimums:

$$\min\left(\frac{x}{y}, \frac{y}{x}\right), \quad \min\left(\frac{x^2}{y^2}, 1\right).$$

It still depends on f and g, as well as an auxiliary function denoted T.

These theorems, which involve various minimum-maximum ratios, are motivated by their deep connections to analysis, operator theory, and the theory of mathematical inequalities. They provide effective tools for estimating integral operators and for establishing precise analytical bounds. Notably, the resulting upper bounds in both theorems are mathematically tractable and can be expressed through simple weighted integral norms with explicit constant factors. These constants are defined in terms of the integrals of the auxiliary functions S and T. The proofs make use of various integral techniques, including the Hölder integral inequality, the Chasles integral relation, the Fubini-Tonelli integral theorem and changes of variables. To illustrate the scope and implications of our results, we present examples, particularly those connected to the gamma and beta functions. These examples demonstrates the versatility of the established inequalities in capturing a wide range of integral behaviors, particularly in contexts where classical special functions play a central role.

The remainder of the article is organized as follows: The first and second theorems are the main subject of Sections 2 and 3, respectively. Section 4 provides a conclusion.

2 First theorem

2.1 Statement and proof

A general and new type of Hardy-Hilbert integral inequality is stated in the theorem below.

Theorem 2.1. *Let $p > 1$, $q = p/(p-1)$, $f, g : (0, +\infty) \to (0, +\infty)$ be two functions, and $S : (1, +\infty) \to (0, +\infty)$ be a function. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty} \left[\max\left(\frac{x^2}{y^2}, 1\right)\right]^{1/p} \left[\max\left(\frac{y^2}{x^2}, 1\right)\right]^{1/q} S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) f(x)g(y)dxdy$$
$$\leq \Omega \left[\int_0^{+\infty} x f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y)dy\right]^{1/q},$$

where

$$\Omega = 2\int_1^{+\infty} S(z)dz, \tag{1}$$

provided that the three integrals composing the upper bounds converge.

Proof of Theorem 2.1. Using a suitable decomposition of the integrand that exploits $1/p + 1/q = 1$

and applying the Hölder integral inequality to the exponents p and q, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q} S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) f^p(x)\right]^{1/p}\\
&\times\left[\max\left(\frac{y^2}{x^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) g^q(y)\right]^{1/q} dxdy\\
&\leq A^{1/p}B^{1/q},
\end{aligned} \tag{2}$$

where

$$A=\int_0^{+\infty}\int_0^{+\infty}\max\left(\frac{x^2}{y^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) f^p(x)dxdy$$

and

$$B=\int_0^{+\infty}\int_0^{+\infty}\max\left(\frac{y^2}{x^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) g^q(y)dxdy.$$

Let us study the exact expressions of A and B in turn.

Concerning A, the Fubini-Tonelli integral theorem gives

$$A=\int_0^{+\infty} f^p(x)\left[\int_0^{+\infty}\max\left(\frac{x^2}{y^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) dy\right]dx.$$

For the central integral, using the Chasles integral relation, the definition of the maximum, and the changes of variables $u=x/y$ and $v=y/x$, we have

$$\begin{aligned}
&\int_0^{+\infty}\max\left(\frac{x^2}{y^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) dy\\
&=\int_0^{x}\max\left(\frac{x^2}{y^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) dy+\int_x^{+\infty}\max\left(\frac{x^2}{y^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) dy\\
&=\int_0^{x}\frac{x^2}{y^2}S\left(\frac{x}{y}\right)dy+\int_x^{+\infty}S\left(\frac{y}{x}\right)dy\\
&=x\int_1^{+\infty}S(u)du+x\int_1^{+\infty}S(v)dv=2x\int_1^{+\infty}S(z)dz=\Omega x,
\end{aligned}$$

where Ω is given by Equation (1).

We therefore have

$$A=\int_0^{+\infty} f^p(x)\times\Omega x dx=\Omega\int_0^{+\infty} x f^p(x)dx, \tag{3}$$

Concerning B, we proceed in a similar way. The Fubini-Tonelli integral theorem gives

$$B=\int_0^{+\infty} g^q(y)\left[\int_0^{+\infty}\max\left(\frac{y^2}{x^2},1\right) S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) dx\right]dy.$$

For the central integral, using the Chasles integral relation, the definition of the maximum, and the changes of variables $u = x/y$ and $v = y/x$, we have

$$\begin{aligned}
&\int_0^{+\infty} \max\left(\frac{y^2}{x^2}, 1\right) S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx \\
&= \int_0^{y} \max\left(\frac{y^2}{x^2}, 1\right) S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx + \int_y^{+\infty} \max\left(\frac{y^2}{x^2}, 1\right) S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx \\
&= \int_0^{y} \frac{y^2}{x^2} S\left(\frac{y}{x}\right) dx + \int_y^{+\infty} S\left(\frac{x}{y}\right) dx \\
&= y\int_1^{+\infty} S(v)dv + y\int_1^{+\infty} S(u)du = 2y\int_1^{+\infty} S(z)dz = \Omega y.
\end{aligned}$$

We therefore have

$$B = \int_0^{+\infty} g^q(y) \times \Omega y dy = \Omega \int_0^{+\infty} y g^q(y) dy. \tag{4}$$

It follows from Equations (2), (3) and (4), and $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} \left[\max\left(\frac{x^2}{y^2}, 1\right)\right]^{1/p} \left[\max\left(\frac{y^2}{x^2}, 1\right)\right]^{1/q} S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) f(x)g(y)dxdy \\
&\leq \left[\Omega \int_0^{+\infty} x f^p(x)dx\right]^{1/p} \left[\Omega \int_0^{+\infty} y g^q(y)dy\right]^{1/q} \\
&= \Omega \left[\int_0^{+\infty} x f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y)dy\right]^{1/q}.
\end{aligned}$$

This completes the proof of Theorem 2.1. □

To the best of our knowledge, this theorem constitutes a new addition to the existing literature on this topic. In a sense, it completes the results in [12, 5, 1, 6, 8, 9, 10, 11, 7, 3]. The auxiliary function S, which defines the constant factor Ω, introduces an additional level of flexibility that we will demonstrate with examples in the next subsection.

Although the constant factor Ω has not been proven to be optimal, it is a serious candidate since it is obtained with the fewest possible steps in the inequalities. A rigorous proof of its optimality remains an open problem.

Let us state an important consequence of the theorem. Using a basic property of the maximum, for any $a, y > 0$, we have

$$\left[\max\left(\frac{x^2}{y^2}, 1\right)\right]^{1/p} \left[\max\left(\frac{y^2}{x^2}, 1\right)\right]^{1/q} \geq 1.$$

This and Theorem 2.1 give

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) f(x)g(y)dxdy \\
&\leq \int_0^{+\infty}\int_0^{+\infty} \left[\max\left(\frac{x^2}{y^2}, 1\right)\right]^{1/p} \left[\max\left(\frac{y^2}{x^2}, 1\right)\right]^{1/q} S\left(\max\left(\frac{x}{y}, \frac{y}{x}\right)\right) f(x)g(y)dxdy \\
&\leq \Omega \left[\int_0^{+\infty} x f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y)dy\right]^{1/q}.
\end{aligned}$$

We thus have derived a simple upper bound for the following general two-dimensional integral:

$$\int_{0}^{+\infty}\int_{0}^{+\infty} S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right) f(x)g(y)dxdy.$$

This also constitutes a new contribution to the article.

2.2 Examples

Four examples of new integral inequalities based on Theorem 2.1 are presented below. Some of these involve the standard beta and gamma functions.

Example 1: If we take $S(x) = x^{-\beta}$, $x \in (1,+\infty)$, with $\beta > 1$, then Theorem 2.1 gives

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]^{-\beta} f(x)g(y)dxdy$$
$$\leq \Omega\left[\int_{0}^{+\infty} x f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} y g^{q}(y)dy\right]^{1/q},$$

where

$$\Omega = 2\int_{1}^{+\infty} S(z)dz = 2\int_{1}^{+\infty} z^{-\beta}dz = \frac{2}{\beta-1}.$$

Example 2: If we take $S(x) = e^{-\alpha x}$, $x \in (1,+\infty)$, with $\alpha > 0$, then Theorem 2.1 gives

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q} e^{-\alpha\max(x/y,y/x)} f(x)g(y)dxdy$$
$$\leq \Omega\left[\int_{0}^{+\infty} x f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} y g^{q}(y)dy\right]^{1/q},$$

where

$$\Omega = 2\int_{1}^{+\infty} S(z)dz = 2\int_{1}^{+\infty} e^{-\alpha z}dz = \frac{2}{\alpha}e^{-\alpha}.$$

Example 3: If we take $S(x) = x^{-\gamma-\tau}/(x-1)^{1-\gamma}$, $x \in (1,+\infty)$, with $\gamma,\tau > 0$, then Theorem 2.1 gives

$$\int_{0}^{+\infty}\int_{0}^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\frac{[\max(x/y,y/x)]^{-\gamma-\tau}}{[\max(x/y,y/x)-1]^{1-\gamma}} f(x)g(y)dxdy$$
$$\leq \Omega\left[\int_{0}^{+\infty} x f^{p}(x)dx\right]^{1/p}\left[\int_{0}^{+\infty} y g^{q}(y)dy\right]^{1/q},$$

where, by the change of variables $w = 1/z$,

$$\Omega = 2\int_{1}^{+\infty} S(z)dz = 2\int_{1}^{+\infty}\frac{z^{-\gamma-\tau}}{(z-1)^{1-\gamma}}dz = 2\int_{0}^{1} w^{\tau-1}(1-w)^{\gamma-1}dw = 2B(\tau,\gamma),$$

and $B(\tau,\gamma)$ denotes the standard beta function at τ and γ. Written in full, the inequality becomes

$$\int_0^{+\infty}\int_0^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\frac{[\max(x/y,y/x)]^{-\gamma-\tau}}{[\max(x/y,y/x)-1]^{1-\gamma}}f(x)g(y)dxdy$$
$$\leq 2B(\tau,\gamma)\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q}.$$

Example 4: If we take $S(x)=e^{1-x}/(x-1)^{1-\lambda}$, $x\in(1,+\infty)$, with $\lambda>0$, then Theorem 2.1 gives

$$\int_0^{+\infty}\int_0^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\frac{1}{[\max(x/y,y/x)-1]^{1-\lambda}}$$
$$\times e^{1-\max(x/y,y/x)}f(x)g(y)dxdy$$
$$\leq \Omega\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q},$$

where, by the change of variables $w=z-1$,

$$\Omega=2\int_1^{+\infty}S(z)dz=2\int_1^{+\infty}\frac{1}{(z-1)^{1-\lambda}}e^{1-z}dz=2\int_0^{+\infty}w^{\lambda-1}e^{-w}dw=2\Gamma(\lambda),$$

and $\Gamma(\lambda)$ denotes the standard gamma function at λ. Written in full, the inequality becomes

$$\int_0^{+\infty}\int_0^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\frac{1}{[\max(x/y,y/x)-1]^{1-\lambda}}$$
$$\times e^{1-\max(x/y,y/x)}f(x)g(y)dxdy$$
$$\leq 2\Gamma(\lambda)\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q}.$$

3 Second theorem

3.1 Statement and proof

The theorem below states a new and general type of Hardy-Hilbert integral inequality. It can be presented as the minimum version of Theorem 2.1. Compared to this result, the constant factor is redefined.

Theorem 3.1. *Let $p>1$, $q=p/(p-1)$, $f,g:(0,+\infty)\to(0,+\infty)$ be two functions, and $T:(0,1)\to(0,+\infty)$ be a function. Then we have*

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)f(x)g(y)dxdy$$
$$\leq \Xi\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q},$$

where

$$\Xi = 2\int_0^1 T(z)dz, \tag{5}$$

provided that the three integrals composing the upper bounds converge.

Proof of Theorem 3.1. Using a suitable decomposition of the integrand that exploits $1/p + 1/q = 1$ and applying the Hölder integral inequality to the exponents p and q, we get

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q} T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right) f(x)g(y)dxdy\\
&=\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)f^p(x)\right]^{1/p}\\
&\times\left[\min\left(\frac{y^2}{x^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)g^q(y)\right]^{1/q}dxdy\\
&\le C^{1/p}D^{1/q},
\end{aligned} \tag{6}$$

where

$$C=\int_0^{+\infty}\int_0^{+\infty}\min\left(\frac{x^2}{y^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)f^p(x)dxdy$$

and

$$D=\int_0^{+\infty}\int_0^{+\infty}\min\left(\frac{y^2}{x^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)g^q(y)dxdy.$$

Let us study the exact expressions of C and D in turn.

Concerning C, the Fubini-Tonelli integral theorem gives

$$C=\int_0^{+\infty}f^p(x)\left[\int_0^{+\infty}\min\left(\frac{x^2}{y^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)dy\right]dx.$$

For the central integral, using the Chasles integral relation, the definition of the minimum, and the changes of variables $u = x/y$ and $v = y/x$, we have

$$\begin{aligned}
&\int_0^{+\infty}\min\left(\frac{x^2}{y^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)dy\\
&=\int_0^{x}\min\left(\frac{x^2}{y^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)dy+\int_x^{+\infty}\min\left(\frac{x^2}{y^2},1\right)T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)dy\\
&=\int_0^{x}T\left(\frac{y}{x}\right)dy+\int_x^{+\infty}\frac{x^2}{y^2}T\left(\frac{x}{y}\right)dy\\
&=x\int_0^1 T(v)dv+x\int_0^1 T(u)du=2x\int_0^1 T(z)dz=\Xi x,
\end{aligned}$$

where Ξ is given by Equation (5).

We therefore have

$$C = \int_0^{+\infty} f^p(x) \times \Xi x dx = \Xi \int_0^{+\infty} x f^p(x) dx. \tag{7}$$

Concerning D, we proceed in a similar way. The Fubini-Tonelli integral theorem gives

$$D = \int_0^{+\infty} g^q(y) \left[\int_0^{+\infty} \min\left(\frac{y^2}{x^2}, 1\right) T\left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx \right] dy.$$

For the central integral, using the Chasles integral relation, the definition of the minimum, and the changes of variables $u = x/y$ and $v = y/x$, we have

$$\begin{aligned}
&\int_0^{+\infty} \min\left(\frac{y^2}{x^2}, 1\right) T\left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx \\
&= \int_0^{y} \min\left(\frac{y^2}{x^2}, 1\right) T\left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx + \int_y^{+\infty} \min\left(\frac{y^2}{x^2}, 1\right) T\left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right) dx \\
&= \int_0^{y} T\left(\frac{x}{y}\right) dx + \int_y^{+\infty} \frac{y^2}{x^2} T\left(\frac{y}{x}\right) dx \\
&= y \int_0^1 T(u) du + y \int_0^1 T(v) dv = 2y \int_0^1 T(z) dz = \Xi y.
\end{aligned}$$

We therefore have

$$D = \int_0^{+\infty} g^q(y) \times \Xi y dy = \Xi \int_0^{+\infty} y g^q(y) dy. \tag{8}$$

It follows from Equations (6), (7) and (8), and $1/p + 1/q = 1$ that

$$\begin{aligned}
&\int_0^{+\infty} \int_0^{+\infty} \left[\min\left(\frac{x^2}{y^2}, 1\right)\right]^{1/p} \left[\min\left(\frac{y^2}{x^2}, 1\right)\right]^{1/q} T\left(\min\left(\frac{x}{y}, \frac{y}{x}\right)\right) f(x) g(y) dx dy \\
&\leq \left[\Xi \int_0^{+\infty} x f^p(x) dx\right]^{1/p} \left[\Xi \int_0^{+\infty} y g^q(y) dy\right]^{1/q} \\
&= \Xi \left[\int_0^{+\infty} x f^p(x) dx\right]^{1/p} \left[\int_0^{+\infty} y g^q(y) dy\right]^{1/q}.
\end{aligned}$$

This completes the proof of Theorem 3.1. □

Like Theorem 2.1, this theorem is a new addition to the existing literature on this topic. The auxiliary function T, which defines the constant factor Ξ, introduces an additional level of flexibility that will be illustrated with examples in the next subsection.

Although the constant factor Ξ has not been proven to be optimal, it is a serious candidate since it is obtained with the fewest possible steps in the inequalities.

3.2 Examples

Four examples of new integral inequalities based on Theorem 3.1 are presented below. Some of these involve the standard beta and gamma functions.

Example 1: If we take $T(x) = x^{\eta}$, $x \in (0,1)$, with $\eta > -1$, then Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\eta} f(x)g(y)dxdy$$
$$\leq \Xi\left[\int_0^{+\infty} xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} yg^q(y)dy\right]^{1/q},$$

where

$$\Xi = 2\int_0^1 T(z)dz = 2\int_0^1 z^{\eta}dz = \frac{2}{\eta+1}.$$

Example 2: If we take $T(x) = \log(1/x)$, $x \in (0,1)$, noting that $1/\min(x/y, y/x) = \max(x/y, y/x)$, then Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\log\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]$$
$$\leq \Xi\left[\int_0^{+\infty} xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} yg^q(y)dy\right]^{1/q},$$

where

$$\Xi = 2\int_0^1 T(z)dz = 2\int_0^1 [-\log(z)]dz = 2.$$

We thus get the elegant inequality

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\log\left[\max\left(\frac{x}{y},\frac{y}{x}\right)\right]$$
$$\leq 2\left[\int_0^{+\infty} xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} yg^q(y)dy\right]^{1/q}.$$

Example 3: If we take $T(x) = x^{\theta-1}(1-x)^{\xi-1}$, $x \in (0,1)$, with $\theta, \xi > 0$, then Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\theta-1}$$
$$\times\left[1-\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\xi-1} f(x)g(y)dxdy$$
$$\leq \Xi\left[\int_0^{+\infty} xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} yg^q(y)dy\right]^{1/q},$$

where

$$\Xi = 2\int_0^1 T(z)dz = 2\int_0^1 z^{\theta-1}(1-z)^{\xi-1}dz = 2B(\theta,\xi).$$

Written in full, the inequality becomes

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\left[\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\theta-1}$$
$$\times\left[1-\min\left(\frac{x}{y},\frac{y}{x}\right)\right]^{\xi-1}f(x)g(y)dxdy$$
$$\leq 2B(\theta,\xi)\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q}.$$

Example 4: If we take $T(x) = x^{-v-1}e^{1-1/x}/(1-x)^{1-v}$, $x \in (0,1)$, with $v > 0$, then Theorem 3.1 gives

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\frac{[\min(x/y,y/x)]^{-v-1}}{[1-\min(x/y,y/x)]^{1-v}}e^{1-\max(x/y,y/x)}$$
$$\leq \Xi\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q},$$

where, by the change of variables $w = 1/z - 1$,

$$\Xi = 2\int_0^1 T(z)dz = 2\int_0^1\frac{z^{-v-1}}{(1-z)^{1-v}}e^{1-1/z}dz = 2\int_0^{+\infty}w^{v-1}e^{-w}dw = 2\Gamma(v).$$

Written in full, the inequality becomes

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}\frac{[\min(x/y,y/x)]^{-v-1}}{[1-\min(x/y,y/x)]^{1-v}}e^{1-\max(x/y,y/x)}$$
$$\leq 2\Gamma(v)\left[\int_0^{+\infty}xf^p(x)dx\right]^{1/p}\left[\int_0^{+\infty}yg^q(y)dy\right]^{1/q}.$$

4 Conclusion

In conclusion, we have established new contributions to the fields of integral inequalities by determining manageable upper bounds for the two following two-dimensional integrals:

$$\int_0^{+\infty}\int_0^{+\infty}\left[\max\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\max\left(\frac{y^2}{x^2},1\right)\right]^{1/q}S\left(\max\left(\frac{x}{y},\frac{y}{x}\right)\right)f(x)g(y)dxdy$$

and

$$\int_0^{+\infty}\int_0^{+\infty}\left[\min\left(\frac{x^2}{y^2},1\right)\right]^{1/p}\left[\min\left(\frac{y^2}{x^2},1\right)\right]^{1/q}T\left(\min\left(\frac{x}{y},\frac{y}{x}\right)\right)f(x)g(y)dxdy.$$

Several examples demonstrate the flexibility of these results, which depend on the different functions chosen for S and T. Two possible extensions of this work would be to study other minimums-maximums versions of these integrals and investigate the optimality of the upper bound obtained. We could also consider three-dimensional variations. These topics will be covered in future articles.

Declarations

Conflicts of interest

The author declares that he has no competing interests.

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References

[1] L. E. Azar, On some extensions of Hardy-Hilbert's inequality and applications, *J. Inequal. Appl.* **2008** (2008), 1–14.

[2] Q. Chen and B. C. Yang, A survey on the study of Hilbert-type inequalities, *J. Inequal. Appl.* **2015** (2015), 1–29.

[3] C. Chesneau, Refining and extending two special Hardy-Hilbert-type integral inequalities, *Ann. Math. Comput. Sci.* **28** (2025), 21–45.

[4] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[5] Y. Li, J. Wu, and B. He, A new Hilbert-type integral inequality and the equivalent form, *Int. J. Math. Math. Sci.* **8** (2006), 1–6.

[6] A. Saglam, H. Yildirim, and M. Z. Sarikaya, Generalization of Hardy-Hilbert's inequality and applications, *Kyungpook Math. J.* **50** (2010), 131–152.

[7] M. Z. Sarikaya and M. S. Bingol, Recent developments of integral inequalities of the Hardy-Hilbert type, *Turkish J. Inequal.* **8** (2024), 43–54.

[8] W. T. Sulaiman, New Hardy-Hilbert's-type integral inequalities, *Int. Math. Forum* **3** (2008), 2139–2147.

[9] W. T. Sulaiman, Hardy-Hilbert's integral inequality in new kinds, *Math. Commun.* **15** (2010), 453–461.

[10] W. T. Sulaiman, New kinds of Hardy-Hilbert's integral inequalities, *Appl. Math. Lett.* **23** (2010), 361–365.

[11] W. T. Sulaiman, An extension of Hardy-Hilbert's integral inequality, *Afr. Diaspora J. Math.* **10** (2010), 66–71.

[12] B. Sun, Best generalization of a Hilbert type inequality, *J. Inequal. Pure Appl. Math.* **7** (2006), 1–7.

[13] B. C. Yang, *Hilbert-Type Integral Inequalities*, Bentham Science Publishers, The United Arab Emirates, 2009.

[14] B. C. Yang, *The Norm of Operator and Hilbert-Type Inequalities*, Science Press, Beijing, 2009.